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1980 J. Phys. A: Math. Gen. 13 L271

(http://iopscience.iop.org/0305-4470/13/8/002)

J. Phys. A: Math. Gen. 13 (1980) L271–L277. Printed in Great Britain

LETTER TO THE EDITOR

Instantons in superfluid ^{3}He

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Received 14 April 1980

Abstract. Using a model quantum mechanical free energy functional, the nucleation rate of superfluid ^{3}He from the A phase to the B phase is discussed.

There has been a great deal of recent interest in the use of finite action solutions of field equations continued to imaginary time (instantons), to investigate quantum mechanical tunnelling between degenerate ground states of quantum field theories. (For reviews see for example Coleman (1977) and Jackiw (1977).) There has also been interest in similar solutions of the field equations when the field theory possesses a metastable state and a stable ground state. These solutions (which have been referred to as 'bounces' by Coleman (1977)) may be used to calculate the nucleation rate from the metastable state to the stable state. These recent calculations are a rediscovery of earlier work in the context of condensed state physics by Langer (1969), Lifshitz and Kagan (1972) and Iordanskii and Finkel'shtein (1972), where the 'bounce' appeared as a critical bubble for nucleation. It has been suggested by Leggett (1978) that these ideas may be useful in the study of nucleation in superfluid ^{3}He.

We wish to discuss the formation of a bubble of stable phase within the metastable phase which is just large enough to expand and convert all the metastable phase to stable phase (a critical bubble). It will be important at sufficiently low temperatures to consider the influence of quantum mechanical fluctuations on the formation of the critical bubble. *A priori*, we expect the critical bubble to be formed by thermal fluctuations at sufficiently high temperatures, by quantum mechanical fluctuations at zero temperature, and by thermally assisted quantum fluctuations at intermediate temperatures. To study these quantum fluctuations we require a quantum free energy functional, such as has been developed to study quantum mechanical phase transitions at zero temperature in the case of magnetic systems (Hertz 1976, Young 1975). A quantum free energy functional may be derived for superfluid ^{3}He by a generalisation of a technique which has been used by Kleinert (1977, 1978) to obtain the usual (non-quantum) Ginzburg–Landau free energy functional. (This technique is very similar to the transformation devised by Stratonovich (1957) and Hubbard (1959) in a different context.) We start from the partition function in terms of the fundamental fields for the ^{3}He atoms.

$$Z = \int \mathscr{D}\psi^{+}\mathscr{D}\psi \exp(-S[\psi]/\hbar), \tag{1}$$

0305-4470/80/080271+07$01.50

where

$$S[\psi]=\int_{-\frac{1}{2}\beta\hbar}^{\frac{1}{2}\beta\hbar}\mathrm{d}\tau\int \mathrm{d}^3x\Big\{\psi^+\Big(\hbar\frac{\partial}{\partial\tau}-\frac{\hbar^2}{2m}\nabla^2-\mu\Big)\psi-\frac{1}{4}g\psi^+\sigma_a\sigma_2\overleftrightarrow{\nabla}\psi^+\cdot\psi\sigma_2\sigma_a\overleftrightarrow{\nabla}\psi\Big\}. \tag{2}$$

In equation (1) the path integral is over fields periodic on the interval $-\frac{1}{2}\beta\hbar\leq\tau<\frac{1}{2}\beta\hbar$, μ is the chemical potential, and only the part of the interaction which produces p-wave, spin–triplet pairing has been retained. If we add to $S[\psi]$ the term

$$\Delta S=(4g)^{-1}\int \mathrm{d}\tau\int \mathrm{d}^3x|k_{\mathrm{F}}^{-1}\Delta_{ai}-g\psi\sigma_2\sigma_a\overleftrightarrow{\nabla}_i\psi|^2, \tag{3}$$

and perform the path integrals over Δ and Δ^+ as well as ψ and ψ^+, we merely multiply Z by a constant. (Because no derivatives of Δ_{ai} occur, Δ_{ai} is not an independent field.) Adding ΔS to $S[\psi]$ eliminates the terms quartic in ψ, and the Gaussian path integral for ψ may be performed. The resulting operator determinant may be evaluated order by order in Δ_{ai}. The result of this calculation is to recast the partition function in terms of Δ_{ai}. Keeping up to quartic order in Δ_{ai}, and neglecting order T_c/T_F, we find

$$Z=\int \mathscr{D}\Delta_{ai}^+\mathscr{D}\Delta_{ai}\exp(-S[\Delta]/\hbar), \tag{4}$$

where

$$S[\Delta]=\int_{-\frac{1}{2}\beta\hbar}^{\frac{1}{2}\beta\hbar}\mathrm{d}\tau\int \mathrm{d}^3x\,\mathscr{L}_{\mathrm{eff}}(\Delta), \tag{5}$$

and

$$\mathscr{L}_{\mathrm{eff}}(\Delta)=\mathscr{L}_B+\mathscr{L}_S+\mathscr{L}_\tau \tag{6}$$

with

$$\mathscr{L}_B=\tfrac{1}{2}(\mathrm{d}n/\mathrm{d}\epsilon)[(T-T_{c1})/T_c]\Delta_{ai}^+\Delta_{ai}+\beta_i\Delta_{ai}\Delta_{ai}\Delta_{bj}^+\Delta_{bj}^++\beta_2\Delta_{ai}^+\Delta_{ai}\Delta_{bj}^+\Delta_{bj}+\beta_3\Delta_{ai}^+\Delta_{bi}^+\Delta_{aj}\Delta_{bj}$$
$$+\beta_4\Delta_{ai}^+\Delta_{bi}\Delta_{bj}^+\Delta_{aj}+\beta_5\Delta_{ai}^+\Delta_{bi}\Delta_{bj}\Delta_{aj}^+, \tag{7}$$

$$\mathscr{L}_S=K_L\,\partial_i\Delta_{ai}^+\,\partial_j\Delta_{aj}+K_T\epsilon_{ijk}\,\partial_j\Delta_{ak}^+\epsilon_{ilm}\,\partial_l\Delta_{am} \tag{8}$$

and

$$\int_{-\frac{1}{2}\beta\hbar}^{\frac{1}{2}\beta\hbar}\mathrm{d}\tau\int \mathrm{d}^3x\,\mathscr{L}_\tau=\frac{1}{2}\frac{\mathrm{d}n}{\mathrm{d}\epsilon}\sum_{n=0}^{\infty}\int\frac{\mathrm{d}^3k}{(2\pi)^3}\frac{1}{\beta\hbar}\Big\{\sum_{p\text{ even}}\frac{\beta\hbar\pi^{-1}|\omega_p|\Delta_{ai}^+(\boldsymbol{k},\omega_p)\Delta_{ai}(\boldsymbol{k},\omega_p)}{(2n+1)[2n+1+\beta\hbar|\omega_p|(2\pi)^{-1}]}\Big\}. \tag{9}$$

In equation (7) we allow strong coupling connections to the β_i so that $\beta_1:\beta_2:\beta_3:\beta_4:\beta_5=-1:2+\delta:2:2-\delta:-2-\delta$ with

$$\beta_3=(21/80)\zeta(3)(\pi k_{\mathrm{B}}T_c)^{-2}(\mathrm{d}n/\mathrm{d}\epsilon), \tag{10}$$

where δ is temperature and pressure dependent. In equation (8)

$$K_L=3K_T=\frac{9}{10}\frac{\mathrm{d}n}{\mathrm{d}\epsilon}\xi_0^2=\frac{9}{10}\frac{\mathrm{d}n}{\mathrm{d}\epsilon}\frac{7\zeta(3)}{48\pi^2}\Big(\frac{\hbar v_{\mathrm{F}}}{k_{\mathrm{B}}T_c}\Big)^2, \tag{11}$$

and in equation (9) there is a summation over Matsubara frequencies

$$\omega_p=p\pi/\beta\hbar \tag{12}$$

with p an even integer, and the Fourier transform

$$\beta\hbar(2\pi)^3\Delta_{ai}(\boldsymbol{x},\tau)=\sum_{p\text{ even}}\int \mathrm{d}^3k\,\exp(-\mathrm{i}\omega_p\tau+\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{x})\Delta_{ai}(\boldsymbol{k},\omega_p) \tag{13}$$

has been made. If equation (9) is rewritten in $(\boldsymbol{x},\tau)$ space, the result is non-local, that is of the form

$$\int_{-\beta\hbar/2}^{\beta\hbar/2}\mathrm{d}\tau\int\mathrm{d}^3x\,\mathscr{L}_\tau=\int_{-\beta\hbar/2}^{\beta\hbar/2}\mathrm{d}\tau\int_{-\beta\hbar/2}^{\beta\hbar/2}\mathrm{d}\tau'\int\mathrm{d}^3x\{\Delta^+_{ai}(\boldsymbol{x},\tau)K(\tau,\tau')\Delta_{ai}(\boldsymbol{x},\tau')\}. \tag{14}$$

If we retain only the $|\omega_p|$ term in equation (9), as might be appropriate in a Ginzburg–Landau expansion,

$$K(\tau,\tau')=-\frac{\pi^2}{16\beta_c\hbar}\frac{\mathrm{d}n}{\mathrm{d}\epsilon}\operatorname{cosec}^2\left[\frac{\pi(\tau'-\tau)}{\beta_c\hbar}\right]. \tag{15}$$

To obtain a model effective Lagrangian which is local, we approximate $K(\tau,\tau')$ by the second derivative of a delta function. Then

$$\mathscr{L}_\tau=\frac{1}{32}(\beta_c\hbar)^2\frac{\mathrm{d}n}{\mathrm{d}\epsilon}\frac{\partial}{\partial\tau}\Delta^+_{ai}(\boldsymbol{x},\tau)\frac{\partial}{\partial\tau}\Delta_{ai}(\boldsymbol{x},\tau). \tag{16}$$

We wish to discuss the formation of a critical bubble of B phase in a metastable A phase using the model effective Lagrangian of equations (6), (7), (8) and (16). In general the wall of such a bubble may involve many components of the 3×3 order parameter Δ_{ai} varying. However, in what follows we use the following simple form of order parameter.

$$\Delta_{ai}=\frac{\Delta(A)}{\sqrt{2}}(1-\alpha)\begin{pmatrix}1&\mathrm{i}&0\\0&0&0\\0&0&0\end{pmatrix}+\frac{\Delta(B)}{\sqrt{3}}\alpha\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix}, \tag{17}$$

with $0\leqslant\alpha\leqslant1$ and $\Delta(\mathrm{A})$ and $\Delta(\mathrm{B})$ the magnitudes of the A and B phase order parameters in the given conditions of temperature and pressure. (Kaul and Kleinert (1980) have used such a form to calculate the surface tension at a planar interface between ^{3}He A and ^{3}He B and have obtained good agreement with experiment.) Substituting (17) into (6), (7), (8) and (16) leads to an effective action of the form

$$S[\alpha]=\frac{\mathrm{d}n}{\mathrm{d}\epsilon}\Delta^2(A)\int_{-\beta\hbar/2}^{\beta\hbar/2}\mathrm{d}\tau\int\mathrm{d}^3x\,\bar{\mathscr{L}}(\alpha), \tag{18}$$

with

$$\bar{\mathscr{L}}(\alpha)=\left(\frac{a\beta_c\hbar}{2\pi}\right)^2\left(\frac{\partial\alpha}{\partial\tau}\right)^2+(c_1\xi_0)^2\left(\frac{\partial\alpha}{\partial x}\right)^2+(c_2\xi_0)^2\left(\frac{\partial\alpha}{\partial y}\right)^2+(c_3\xi_0)^2\left(\frac{\partial\alpha}{\partial z}\right)^2+\frac{T_c-T}{4T_c}f(\alpha), \tag{19}$$

where

$$f(\alpha)=A\alpha^2(1-\alpha)^2+\eta\alpha^2(2\alpha-3), \tag{20}$$

$$\eta=1-\frac{F(\mathrm{B})}{F(\mathrm{A})}. \tag{21}$$

$F(\mathrm{B})$ and $F(\mathrm{A})$ are the free energies of the B and A phases and A, a, c_1, c_2, and c_3 are pure numbers of order 1. With the model we have used we find

$$A = \frac{35-\delta}{5+\delta} - 8\left(\frac{2-\delta}{10+2\delta}\right)^{1/2}$$

using the values (10) for $\beta_1, \ldots \beta_5$. At the A–B transition $\delta = \frac{1}{4}$, so we expect $A \approx 3{\cdot}3$. In the same model we find

$$a^2 = \frac{\pi^2}{8}\left(1 - \left(\frac{2}{3}\right)^{1/2} \frac{\Delta(\mathrm{B})}{\Delta(\mathrm{A})} + \frac{\Delta^2(\mathrm{B})}{\Delta^2(\mathrm{A})}\right)$$

$$c_1^2 = \frac{3}{10}\left(2 - 6^{1/2} \frac{\Delta(\mathrm{B})}{\Delta(\mathrm{A})} + \frac{5}{3} \frac{\Delta^2(\mathrm{B})}{\Delta^2(\mathrm{A})}\right)$$

$$c_2^2 = \frac{3}{10}\left(2 - \left(\frac{2}{3}\right)^{1/2} \frac{\Delta(\mathrm{B})}{\Delta(\mathrm{A})} + \frac{5}{3} \frac{\Delta^2(\mathrm{B})}{\Delta^2(\mathrm{A})}\right)$$

$$c_3^2 = \frac{3}{10}\left(1 - \left(\frac{2}{3}\right)^{1/2} \frac{\Delta(\mathrm{B})}{\Delta(\mathrm{A})} + \frac{5}{3} \frac{\Delta^2(\mathrm{B})}{\Delta^2(\mathrm{A})}\right).$$

In general $\Delta^2(\mathrm{B})/\Delta^2(\mathrm{A}) = (6-3\delta)/5+\delta$, so near the transition we expect $ac_1c_2c_3 \approx 0{\cdot}50$. It is convenient to absorb a factor $a\beta_c\hbar/2\pi$ into the definition of τ and factors $c_1\xi_0$, $c_2\xi_0$ and $c_3\xi_0$ into the definitions of x, y, z to obtain dimensionless variables. Then,

$$S[\alpha] = \frac{\mathrm{d}n}{\mathrm{d}\epsilon}\,\Delta^2(A)(ac_1c_2c_3)\xi_0^3\beta_c\hbar(2\pi)^{-1}\int_{-\beta/2\beta_0}^{\beta/2\beta_0}\mathrm{d}\tau\int \mathrm{d}^3x\,\mathscr{L}(\alpha), \tag{22}$$

with

$$\beta_0 = a\beta_c/2\pi \tag{23}$$

and

$$\mathscr{L}(\alpha) = \left(\frac{\partial\alpha}{\partial\tau}\right)^2 + (\nabla\alpha)^2 + \frac{T_c - T}{4T_c} f(\alpha) \tag{24}$$

and $f(\alpha)$ as in equation (20).

The nucleation rate per unit volume of the metastable A phase into the B phase is now

$$I = I_0 \exp\left(-S[\bar\alpha]/\hbar\right) \tag{25}$$

where $\bar\alpha(\boldsymbol{x}, \tau)$ is the solution of the Euler–Lagrange equations corresponding to the critical bubble, and I_0 is some fundamental rate (Langer 1969, Lifshitz and Kagan 1972, Coleman 1977 and references therein). We concentrate on the exponential factor in equation (21) which is more sensitive to the form of the critical bubble than I_0. Even at $T = 0$ it is not possible to solve the Euler–Lagrange equations exactly for the quantum critical bubble (see Coleman 1977). In order to estimate the quantum bubble at finite T we use the following parametrisation, in which R is related to the radius of the critical bubble, and λ^{-1} to the thickness of the bubble wall:

$$\bar\alpha(r, \tau) = \begin{cases} 1 - \lambda(\rho - R) & \text{for } R \leqslant \rho \leqslant R + \lambda^{-1} \\ 1 & \text{for } \rho < R \\ 0 & \text{for } \rho > R + \lambda^{-1} \end{cases} \tag{26}$$

where

$$\rho = (r^2 + \tau^2)^{1/2}. \tag{27}$$

If we consider bubbles of negligible thickness, then for $\beta/2\beta_0 > R$ discrete bubbles in τ space are periodically repeated, and for $\beta/2\beta_0 < R$ the bubbles have coalesced in τ space (see figures 1 and 2). We shall refer to the temperature, T_B, at which $\beta/2\beta_0 = R$ as the 'bubble coalescence' temperature. When (26) is inserted in (22), the thin-wall approximation $R \gg \lambda^{-1}$ is made, and the resulting effective action is made stationary under variations of λ and R, we find for $T < T_B$,

$$\lambda = [A(T_c - T)/120T_c]^{1/2}, \tag{28}$$

$$R = 24T_c\lambda/(T_c - T)\eta \tag{29}$$

and

$$\int_{-\beta/2\beta_0}^{\beta/2\beta_0} d\tau \int d^3x\, \mathscr{L}(\tilde{\alpha}) = \frac{24\pi^2}{25}\frac{A^2}{\eta^3}\frac{T_c}{T_c - T}. \tag{30}$$

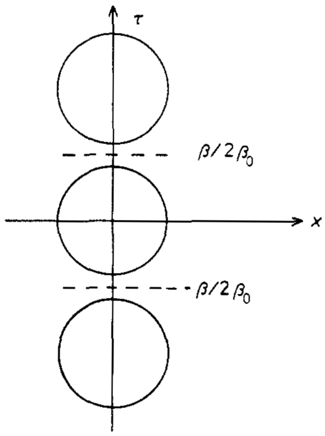

Figure 1. Critical bubble for $T < T_B$.

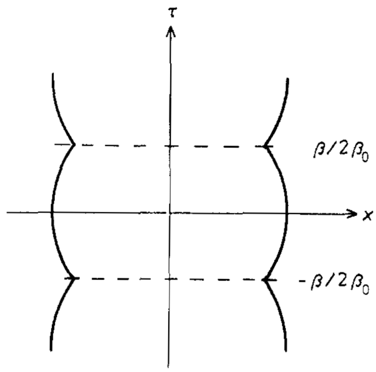

Figure 2. Critical bubble for $T > T_B$.

For $T > T_B$, the results are that λ is given by equation (28) as before, R is the solution of

$$\frac{T_c - T}{4T_c}\eta R\left[\frac{b}{R^2}(R^2 - b^2)^{1/2} + \sin^{-1}(b/R)\right] = 2\lambda\left[\frac{b}{R^2}(R^2 - b^2)^{1/2} + 3\sin^{-1}(b/R)\right], \tag{31}$$

and

$$\int_{-\beta/2\beta_0}^{\beta/2\beta_0} d\tau \int d^3x\, \mathscr{L}(\tilde{\alpha}) = -\frac{T_c - T}{16T_c}\eta[\tfrac{2}{3}b(R^2 - b^2)^{3/2} + bR^2(R^2 - b^2)^{1/2} + R^4\sin^{-1}(b/R)]$$
$$+ 2\lambda R[b(R^2 - b^2)^{1/2} + R^2\sin^{-1}(b/R)], \tag{32}$$

where

$$b = \beta/2\beta_0. \tag{33}$$

The thermal critical bubble $\tilde{\alpha}(r)$, which has no τ dependence, may be estimated using the parameterisation

$$\tilde{\alpha}(r) = \begin{cases} 1 - \lambda(r - R) & \text{for } R \leqslant r \leqslant R + \lambda^{-1} \\ 1 & \text{for } r < R \\ 0 & \text{for } r > R + \lambda^{-1}. \end{cases} \tag{34}$$

In this case a similar calculation yields

$$R = 16T_c\lambda/(T_c - T)\eta, \tag{35}$$

where λ is still given by (28), and

$$\int_{-\beta/2\beta_0}^{\beta/2\beta_0} \mathrm{d}\tau \int \mathrm{d}^3x\, \mathscr{L}(\tilde{\alpha}) = \frac{512}{3}\pi\left(\frac{T_c}{T_c - T}\right)^{1/2} \frac{\beta}{2\beta_0\eta^2}\left(\frac{A}{30}\right)^{3/2} \tag{36}$$

Comparing (30) and (36) we see that $S[\tilde{\alpha}] = S[\tilde{\alpha}]$ when

$$\frac{\beta}{2\beta_0} = \frac{81\pi}{16\eta}\left[\frac{AT_c}{30(T_c - T)}\right]^{1/2}. \tag{37}$$

At the temperature, T_{TH}, given by equation (37) the nucleation rates due to quantum and thermal bubbles are equal. For $T > T_{TH}$, nucleation proceeds by thermal bubbles, and for $T < T_{TH}$ by quantum bubbles. Noticing that the 'bubble coalescence' temperature T_B is given by

$$\frac{\beta}{2\beta_0} = \frac{12}{\eta}\left[\frac{AT_c}{30(T_c - T)}\right]^{1/2}, \tag{38}$$

it follows (assuming that A is temperature independent) that

$$\frac{T_B}{T_{TH}} = \frac{27\pi}{64}\left(\frac{T_c - T_B}{T_c - T_{TH}}\right)^{1/2}, \tag{39}$$

which implies that $T_B > T_{TH}$. Thus, when quantum bubbles are responsible for nucleation the effective action is always given by (30), and never by (32). (We have checked that higher periodicity of the quantum bubble in τ leads to a smaller nucleation rate at those temperatures at which quantum nucleation is more important than thermal nucleation.)

Taking

$$\eta \simeq 0{\cdot}36\,\frac{T_c - T}{T_c} \tag{40}$$

near the polycritical point, we estimate from (37), taking $A \approx 3{\cdot}3$, that

$$T_{TH} \approx 0{\cdot}16T_c. \tag{41}$$

Since the pre-exponential factor I_0 in equation (25) is of order $(\beta_c\hbar)^{-1}\xi_0^{-3}(S[\tilde{\alpha}]/2\pi\hbar)^2$ (see Coleman 1977 and references therein), we also estimate

$$I \approx 10^{-2\times10^4}\,\mathrm{cm}^{-3}\,\mathrm{s}^{-1}. \tag{42}$$

This is very much smaller than the experimental nucleation rate. To obtain the experimental rate, $S[\tilde{\alpha}]$ would need to be three orders of magnitude smaller. This could happen if the constant A, which is a measure of the barrier between the A phase and B phase minima, were much smaller than we have estimated. However, it is hard to see how this could be compatible with the experimental success of estimates of the surface tension at a ^{3}He A to ^{3}He B boundary (see for example Kaul and Kleinert 1980).

Another possibility is that the constants a, c_1, c_2, c_3 are not in fact of order unity. Suppose, for some reason, that the coefficient of $(\partial\alpha/\partial\tau)^2$ in (19) were $(a'\beta_F\hbar/2\pi)^2$ with a' of order unity. This would have the effect of multiplying a, and thereby $S[\tilde{\alpha}]$, by a factor $\beta_F/\beta_c \approx 10^{-3}$. We can think of no reason why this should be so. Alternatively, it might be that replacing the non-local $K(\tau, \tau')$ of equation (15) by the local form of equation (16) has led to a gross overestimate of the energy of quantum fluctuations. Finally, it may be that dissipation through coupling to the normal fluid is playing an important role. (Leggett (1978) has stressed this danger.) Certainly, if the characteristic frequency of excitations is comparable with the characteristic frequency for tunnelling across the barrier between the A phase and the B phase, then this effect is important. A completely new formalism would then be required to calculate the nucleation rate, rather than the calculation based on the imaginary part of the free energy adopted here (and by Langer 1967, Lifshitz and Kagan 1972 and subsequent authors).

Acknowledgments

We would like to thank A J Leggett, A Caldeira and C Harris for helpful discussions. This research was supported in part by the Science Research Council under grant number GR/BO 1 623.

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